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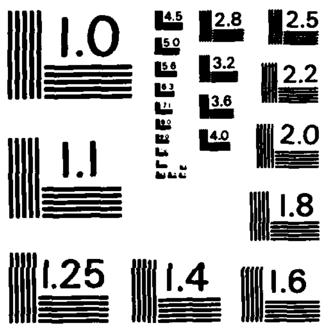

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OPTIMALITY ROBUSTNESS OF TESTS IN
TWO POPULATION PROBLEMS*

by

Takeaki Kariya
Hitotsubashi University

and

Bimal K. Sinha
University of Pittsburgh

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University of Pittsburgh
515 Thackeray Hall
Pittsburgh, PA 15260

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1. Introduction.

Recently a lot of work has been done in the area of robustness of multivariate tests. Robustness typically carries three meanings: null, nonnull, and optimality. Roughly speaking, a test is defined to be null robust if its null distribution remains the same for a class of distributions including the distribution under which the test is derived. Similarly we define the nonnull robustness of a test by the invariance of its nonnull distribution in a class of distributions including the underlying distribution. Finally, a test is called optimality robust if an optimal property the test enjoys can be extended to a class of distributions including the distribution under which the optimality holds. In a series of articles published by the present authors, the various relationships connecting these robustness concepts have been clarified, tools for establishing robustness of many multivariate tests have been developed and extensively used (see e.g. Kariya (1977, 1980, 1981a, 1981b), Kariya and Sinha (1985), Sinha (1984), Sinha and Giri (1985), Sinha and Das (1985)). It turns out that in all the cases considered so far optimality robustness along with null robustness hold for many familiar tests for a broad class of nonnormal distributions. However, it should be noted that in these approaches nonnormality at the sacrifice of independence has been considered. Therefore, in dealing with independent populations, this approach is not quite natural.

In this paper keeping (between) independence of the two populations we consider some testing problems involving location and scale parameters under some conditions on the underlying distributions. To be specific, in Section 2 we have considered the problem of testing the equality of two location parameters without any scale parameter and shown that the test based on the difference between the two sample means is conditional UMPI, conditionally given two ancillary statistics. However, it is not unconditionally so unless either the populations are normal or there is only one observation from each population. In Section 3 we consider the problem of testing the equality of two scale parameters with or without the presence of location parameters and prove that the standard normal-theory UMPI F-test is optimality robust.

In Sections 4 and 5 analogous results have been obtained in association with standard tests considered in life-testing models (exponential distribution).

It should be pointed out that in these tests null robustness does not hold mainly because of independence.

As a technical tool, Wijsman's (1967) representation theorem is used.

2. Tests of Equality of Two Location Parameters (Non-Normal Case).

Let $\underset{\sim}{X}^*$: $(n_1+1)\times 1$ and $\underset{\sim}{Y}^*$: $(n_2+1)\times 1$ be independent random vectors with pdfs (with respect to Lebesgue measure)

$$(2.1)\quad f_{\theta_1}(\underset{\sim}{x}^*) = q_1[(\underset{\sim}{x}^*-\theta_1\underset{\sim}{e}_1)'(\underset{\sim}{x}^*-\theta_1\underset{\sim}{e}_1)],\ f_{\theta_2}(\underset{\sim}{y}^*) = q_2[(\underset{\sim}{y}^*-\theta_2\underset{\sim}{e}_2)'(\underset{\sim}{y}^*-\theta_2\underset{\sim}{e})]$$

where $\underset{\sim}{x}^* \in R^{n_1+1}$, $\underset{\sim}{y}^* \in R^{n_2+1}$, $\underset{\sim}{e}_1 = (1,\ldots,1)' \in R^{n_1+1}$, $\underset{\sim}{e}_2 = (1,\ldots,1)' \in R^{n_2+1}$, $\theta_1, \theta_2 \in R$ and q_1 and q_2 are pdfs over R^{n_1+1} and R^{n_2+1} respectively. Making a suitable orthogonal transformation, without any loss of generality, we can write $\underset{\sim}{x}^* = (x, \underset{\sim}{x}_1')'$, $\underset{\sim}{y}^* = (y, \underset{\sim}{y}_1')'$ where $\underset{\sim}{x}_1 \in R^{n_1}$, $\underset{\sim}{y}_1 \in R^{n_2}$ and assume that the pdfs are

$$(2.2)\qquad f_{\theta_1}(\underset{\sim}{x}^*) = q_1[(x-\theta_1)^2 + \underset{\sim}{x}_1'\underset{\sim}{x}_1],\ f_{\theta_2}(\underset{\sim}{y}^*) = q_2[(y-\theta_2)^2 + \underset{\sim}{y}_1'\underset{\sim}{y}_1].$$

The problem of testing H_0: $\theta_1 = \theta_2$ vs H_1: $\theta_1 > \theta_2$ remains invariant under the group G of transformations, $G = G_1 \times G_2$, where G_1 is the group of translations: $x \to x+c$, $y \to y+c$, $-\infty < c < \infty$, and G_2 is the group of orthogonal transformations: $O(n_1) \times O(n_2)$ acting on $\underset{\sim}{x}_1$ and $\underset{\sim}{y}_1$ as: $\underset{\sim}{x}_1 \to \Gamma_1\underset{\sim}{x}_1$, $\underset{\sim}{y}_1 \to \Gamma_2\underset{\sim}{y}_1$, $\Gamma_i \in O(n_i)$, $i = 1,2$.

Clearly $T = (x-y,\ \underset{\sim}{x}_1'\underset{\sim}{x}_1,\ \underset{\sim}{y}_1'\underset{\sim}{y}_1)$ is a maximal invariant. Using Wijsman's (1967) representation theorem, the ratio of the pdfs of T under nonnull to null is obtained as

$$(2.3)\quad R = dP_{H_1}^{T}/dP_{H_0}^{T} = \frac{\int_{-\infty}^{\infty} q_1[(x+c-\theta_1)^2+\underset{\sim}{x}_1'\underset{\sim}{x}_1]\,q_2[(y+c-\theta_2)^2+\underset{\sim}{y}_1'\underset{\sim}{y}_1]\,dc}{\int_{-\infty}^{\infty} q_1[(x+c-\theta_1)^2+\underset{\sim}{x}_1'\underset{\sim}{x}_1]\,q_2[(y+c-\theta_1)^2+\underset{\sim}{y}_1'\underset{\sim}{y}_1]\,dc}$$

$$= \frac{\int_{-\infty}^{\infty} q_1[c^2+\underset{\sim}{x}_1'\underset{\sim}{x}_1]\,q_2[(c-z+\delta)^2+\underset{\sim}{y}_1'\underset{\sim}{y}_1]\,dc}{\int_{-\infty}^{\infty} q_1[c^2+\underset{\sim}{x}_1'\underset{\sim}{x}_1]\,q_2[(c-z)^2+\underset{\sim}{y}_1'\underset{\sim}{y}_1]\,dc},\quad z = x-y,\ \delta = \theta-\theta_2$$

$$= \frac{\int_{-\infty}^{\infty} q_1[(c-\delta)^2+\underset{\sim}{x}_1'\underset{\sim}{x}_1]\,q_2[(c-z)^2+\underset{\sim}{y}_1'\underset{\sim}{y}_1]\,dc}{\int_{-\infty}^{\infty} q_1[c^2+\underset{\sim}{x}_1'\underset{\sim}{x}_1]\,q_2[(c-z)^2+\underset{\sim}{y}_1'\underset{\sim}{y}_1]\,dc}.$$

To derive an optimum invariate test based on R, note that, conditionally given the two ancillary statistics $\underset{\sim}{x}_1'\underset{\sim}{x}_1$ and $\underset{\sim}{y}_1'\underset{\sim}{y}_1$, the ratio of the conditional pdfs of (x-y) under nonnull to null is precisely R. However, from (2.3), R can be expressed as

$$(2.4)\qquad R = E\{q_1[(c-\delta)^2+\underset{\sim}{x}_1'\underset{\sim}{x}_1]/q_1[c^2+\underset{\sim}{x}_1'\underset{\sim}{x}_1]\}$$

where E stands for expectation with respect to c having the pdf

$$(2.5)\qquad c \sim q_1[c^2+\underset{\sim}{x}_1'\underset{\sim}{x}_1]\,q_2[(c-z)^2+\underset{\sim}{y}_1'\underset{\sim}{y}_1]\Big/\int_{-\infty}^{\infty} q_1[c^2+\underset{\sim}{x}_1'\underset{\sim}{x}_1]\,q_2[(c-z)^2+\underset{\sim}{y}_1'\underset{\sim}{y}_1]\,dc.$$

We now make the following assumption.

Assumption 2.1. $q_1, q_2 \in Q = \{q: \frac{q[(c-a)^2+t]}{q[c^2+t]} \uparrow c$, for every $a > 0$, $t > 0\}$.

It then follows that the family of distributions of c in (2.5), generated by the parameter z, keeping $\underset{\sim}{x}_1'\underset{\sim}{x}_1$ and $\underset{\sim}{y}_1'\underset{\sim}{y}_1$ fixed, has MLR property in c by the assumption $q_2 \in Q$ and hence R is nondecreasing in z by the assumption $q_1 \in Q$ (Lehmann (1959), p. 74). We have, therefore, proved the following.

Theorem 2.1. For testing H_0: $\theta_1 = \theta_2$ vs H_1: $\theta_1 > \theta_2$ in the model (2.2), the test with a critical region $z > k$ is UMPI in the class of conditional level-α tests, conditionally given the two ancillary statistics $\underset{\sim}{x}_1'\underset{\sim}{x}_1$ and $\underset{\sim}{y}_1'\underset{\sim}{y}_1$, whenever $q_1, q_2 \in Q$.

Remark 2.1. It is clear from (2.3) that when both q_1 and q_2 are normal, $\underset{\sim}{x}_1'\underset{\sim}{x}_1$ and $\underset{\sim}{y}_1'\underset{\sim}{y}_1$ can be ignored and the above test is UMPI unconditionally. For nonnormal q's, sufficiency-invariance reduces $(x, y, \underset{\sim}{x}_1'\underset{\sim}{x}_1, \underset{\sim}{y}_1'\underset{\sim}{y}_1)$ to $(x-y, \underset{\sim}{x}_1'\underset{\sim}{x}_1, \underset{\sim}{y}_1'\underset{\sim}{y}_1)$ and the above *conditional argument* is necessary unless $n_1 = n_2 = 0$. This means the test is UMPI when there is only one observation from each population and $q_1, q_2 \in Q$.

Remark 2.2. If $q \in Q$ is differentiable, Q is equivalent to

$$(2.6) \qquad \tilde{Q} = \{q: \left(\frac{cq'(c^2+t)}{q(c^2+t)}\right) \downarrow c, \quad \forall\ t \geq 0\}.$$

An example of a $q \in \tilde{Q}$ is provided by $q(u) = e^{-u^r}$, $r \geq \frac{1}{2}$. It may be remarked that a t-density $\notin$ Q.

Remark 2.3. Our attempt to derive a UMPI or even a LBI test when q_1 and q_2 involve an unknown common scale parameter is not successful unless q_1 and q_2 are normal. It is possible that the familiar Fisher's t-test for this problem based on $(x-y)/(\underset{\sim}{x}_1'\underset{\sim}{x}_1+\underset{\sim}{y}_1'\underset{\sim}{y}_1)^{1/2}$ is optimum only for normal densities.

Remark 2.4. The null pdf of Z is given by

(2.7) $$dP^{Z}_{H_0}/dz = \int_{-\infty}^{\infty} \tilde{q}_1(c^2)\tilde{q}_2((c-z)^2)dc$$

where $\tilde{q}_1(c^2) = \int_{R^{n_1}} q_1(c^2+\underset{\sim}{x}_1'\underset{\sim}{x}_1)d\underset{\sim}{x}_1$, $\tilde{q}_2((c-z)^2) = \int_{R^{n_2}} q_2((c-z)^2+\underset{\sim}{y}_1'\underset{\sim}{y}_1)d\underset{\sim}{y}_1$. Thus the null robustness of the test does not hold.

3. Test of Equality of Two Scale Parameters (Non-Normal Case).

Let $\underset{\sim}{x}_i$ be $n_i \times 1$ random vector with pdf (with respect to Lebesgue measure)

(3.1) $$f_i(\underset{\sim}{x}_i) = \sigma_i^{-n_i} q_i((\underset{\sim}{x}_i - \mu_i\underset{\sim}{e}_i)'(\underset{\sim}{x}_i - \mu_i\underset{\sim}{e}_i)/\sigma_i^2) \quad (i=1,2),$$

where $\mu_i \in R$, $\underset{\sim}{e}_i = (1,\ldots,1)' \in R^{n_i}$ and $\underset{\sim}{x}_1 \in R^{n_1}$ and $\underset{\sim}{x}_2 \in R^{n_2}$ are independent. Here we consider the problem of testing H: $\sigma_1^2 = \sigma_2^2$ vs H_1: $\sigma_1^2 > \sigma_2^2$. Of course, if x_i's are both normal, then the F-test is UMPI. We shall show it is also UMPI in the above situation under a mild condition on q_1, q_2. This implies an optimality robustness of the F-test. Without essential loss of generality, assume $\mu_1 = \mu_2 = 0$. Then the problem remains invariant under the transformation $\underset{\sim}{x}_i \to c\underset{\sim}{x}_i$ $(i=1,2)$, where $c \in G = (0,\infty)$. Then the ratio R of the pdfs of a maximal invariant under nonnull to null is given by $R = N/D$, where

(3.2) $$N \equiv N(\sigma_1,\sigma_2) = \int_G [\Pi_{i=1}^2 \sigma_i^{-n_i} q_i(c^2\underset{\sim}{x}_i'\underset{\sim}{x}_i/\sigma_i^2)]c^{n-1}dc$$

$n = n_1 + n_2$ and $D = N(\sigma_1,\sigma_1)$. In (3.2), transforming c into $c(\underset{\sim}{x}_2'\underset{\sim}{x}_2)^{1/2}/\sigma_2$, N becomes

(3.3) $$N_\gamma = (\underset{\sim}{x}_2'\underset{\sim}{x}_2)^{-n/2}\gamma^{n_1/2}\int_0^\infty q_1(\gamma t c^2)q_2(c^2)c^{n-1}dc$$

where $\gamma = \sigma_2^2/\sigma_1^2$ and $t = \underset{\sim}{x}_1'\underset{\sim}{x}_1/\underset{\sim}{x}_2'\underset{\sim}{x}_2$. Note $D = N_1$. Further let $\mathcal{Q}_n$ be the class of functions from $[0,\infty)$ into $[0,\infty)$ such that for any $q \in \mathcal{Q}_n$, $\int_{R^n} q(\underset{\sim}{x}'\underset{\sim}{x})d\underset{\sim}{x} = 1$ and $q(\beta u)/q(u)$ is increasing in u for any $0 < \beta < 1$. Then we obtain

<u>Theorem 3.1</u>. The test based on $t = \underset{\sim}{x}_1'\underset{\sim}{x}_1/\underset{\sim}{x}_2'\underset{\sim}{x}_2 > k$ is UMPI for testing $\sigma_1^2 = \sigma_2^2$ vs $\sigma_1^2 > \sigma_2^2$ under the pdf's in (3.1) for any $q_1 \in \mathcal{Q}_{n_1}$, $q_2 \in \mathcal{Q}_{n_2}$. But the null distribution of t depends on q_1 and q_2.

<u>Proof</u>. Clearly t is a maximal invariant under G. Under the null hypothesis, the pdf of t is directly shown to be

$$(3.4) \qquad dP_H^t = at^{(n-2)/2}\int_0^\infty q_1(tc^2)q_2(c^2)c^{n-1}dc$$

where a is a constant depending on n_1 and n_2 only and P_H^t is the distribution of t under H_0: $\sigma_1^2 = \sigma_2^2$. Hence from $R = dP_{H_1}^t/dP_{H_0}^t = N_\gamma/N_1$ with N_γ in (3.3) and (3.4), the pdf of t under (σ_1,σ_2) is given by

$$(3.5) \qquad h(t|\gamma)dt = RdP_H^t = at^{(n-2)/2} \qquad \gamma^{n_1/2}[\int_0^\infty q_1(\gamma tc^2)q_2(c^2)c^{n-1}dc]dt,$$

so $h(t|\gamma)/h(t|1) = R$. Further R in this sense is expressed as

$$(3.6) \qquad R = \gamma^{n_1/2}E[q_1(\gamma c^2)/q_1(c^2)]$$

where the expectation is taken with respect to the pdf

$$(3.7) \qquad k_t(c^2) = q_1(c^2)q_2(c^2/t)c^{n-1}/\int_0^\infty q_1(c^2)q_2(c^2/t)c^{n-1}dc.$$

Here we show that $k_t(c^2)$ with $t > 0$ has a monotone likelihood ratio property in c^2. In fact, since $q_2 \in \mathcal{Q}_{n_2}$, $k_{t_2}(c^2)/k_{t_1}(c^2) \propto q_2(c^2/t_2)/q_2(c^2/t_1)$ is increasing in c^2 for $t_2 > t_1$. Therefore, since $q_1(\gamma c^2)/q_1(c^2)$ in the inside of the expectation of (3.6) is increasing in c^2 by assumption, its expectation is increasing in t (see Lehmann (1959) p. 74). This implies R in (3.6) is an increasing function of t. Consequently by the Neyman-Pearson Lemma, the test based on $t > k$ is UMP.

Since t does not depend on q_i's, it is UMPI for all $q_1 \in \mathcal{Q}_{n_1}$, $q_2 \in \mathcal{Q}_{n_2}$. But the null distribution of t in (3.4) cannot be free from q_i's. This proves our result.

Remark 3.1. If $\mu_1 = \mu_2 = 0$ is not assumed, the increasingness of $q(\beta u)/q(u)$ is simply replaced by that of $\bar{q}(\beta u)/\bar{q}(u)$, where $\bar{q}(u) = \int_{-\infty}^{\infty} q(u+x^2)dx$.

Remark 3.2. A crucial distinction of our result from Corollary 4.3 in Kariya (1981) lies in assuming the independence of $\underset{\sim}{x}_1$ and $\underset{\sim}{x}_2$.

Remark 3.3. The null distribution of t is given by (3.5) with $\gamma = 1$. Hence for specific q_1 and q_2, significance points of t are obtained from it.

Remark 3.4. If q is differentiable, a necessary and sufficient condition for $q(\beta u)/q(u)$ to be increasing in u for $0 < \beta < 1$ is given by $q'(u) < 0$ and $\frac{uq'(u)}{q(x)} \downarrow u$. This condition is satisfied for a multivariate t-distribution and also for more general normal mixtures.

4. Test of Equality of Two Location Parameters (Non-Exponential Case).

Let $\underset{\sim}{X}$: $n_1 \times 1$ and $\underset{\sim}{Y}$: $n_2 \times 1$ be independent random vectors with pdfs (with respect to Lebestue measure)

$$(4.1) \qquad f_{\theta_1}(\underset{\sim}{x}) = q_1[\sum_1^{n_1}(x_i-\theta_1)], \; f_{\theta_2}(\underset{\sim}{y}) = q_2[\sum_1^{n_2}(y_i-\theta_2)]$$

$$x_i \geq \theta_1,\; y_i \geq \theta_2,\; \forall i;\; \underset{\sim}{x} \in R^{n_1},\; \underset{\sim}{y} \in R^{n_2},\; \theta_1,\theta_2 \in R.$$

The problem of testing H_0: $\theta_1 = \theta_2$ vs H_1: $\theta_1 > \theta_2$ remains invariant under the group G of translations acting as $x_i \to x_i+c$, $y_i \to y_i+c$, $i = 1,\ldots,n$, $\theta_1 \to \theta_1+c$, $\theta_2 \to \theta_2+c$, $-\infty < c < \infty$. Let $x_{(1)} = \min x_i, y_{(1)} = \min y_i$, $t_1 = \sum_1^{n_1}(x_i-x_{(1)})$, $t_2 = \sum_1^{n_2}(y_i-y_{(1)})$. Applying Wijsman's (1967) theorem, the ratio R of the pdfs of a maximal invariant $T = (x_{(1)}-y_{(1)}, t_1,t_2)$ under nonnull to null is obtained as

$$(4.2)\quad R = \frac{dP^T_{H_1}}{dP^T_{H_0}} = \frac{\int_{c\geq\max(\theta_1-x_{(1)},\theta_2-y_{(1)})} q_1[\sum_1^{n_1}(x_i+c-\theta_1)]q_2[\sum_1^{n_2}(y_i+c-\theta_2)]dc}{\int_{c\geq\max(\theta_1-x_{(1)},\theta_1-y_{(1)})} q_1[\sum_1^{n_1}(x_i+c-\theta_1)]q_2[\sum_1^{n_2}(y_i+c-\theta_1)]dc}$$

$$= \frac{\int_{c\geq\max(\theta_1-x_{(1)},\theta_2-y_{(1)})} q_1[t_1+n_1(x_{(1)}+c-\theta_1)]q_2[t_2+n_2(y_{(1)}+c-\theta_2)]dc}{\int_{c\geq\max[\theta_1-x_{(1)},\theta_1-y_{(1)}]} q_1[t_1+n_1(x_{(1)}+c-\theta_1)]q_2[t_2+n_2(y_{(1)}+c-\theta_1)]dc}$$

$$= \frac{\int_{c\geq\max(0,z-\delta)} q_1[t_1+c]q_2[t_2+\frac{n_2}{n_1}(c-z+\delta)]dc}{\int_{c\geq\max(0,z)} q_1[t_1+c]q_2[t_2+\frac{n_2}{n_1}(c-z)]dc}, \quad z=x_{(1)}-y_{(1)},\ \delta=\theta_1-\theta_2$$

$$= \frac{\int_{c\geq\max(\delta,z)} q_1[t_1+c-\delta]q_2[t_2+\frac{n_2}{n_1}(c-z)]dc}{\int_{c>\max(0,z)} q_1[t_1+c]q_2[t_2+\frac{n_2}{n_1}(c-z)]dc}.$$

To derive an optimum invariant test based on R, note that, conditionally given the two ancillary statistics t_1 and t_2, the ratio of the conditionalpdfs of z under nonnull to null is precisely R. Moreover, from (4.2), R can be expressed as

$$(4.3)\qquad R = E\{q_1[(t_1+c-\delta)]/q_1[t_1+c]\}\cdot I_{[c>\delta]}$$

where I is the indicator function and E above stands for expectation with respect to c having the pdf,

$$(4.4)\quad c \sim q_1[t_1+c]q_2+\frac{n_2}{n_1}(c-z)]/\int_{c>\max(0,z)} q_1[t_1+c]q_2[t_2+\frac{n_2}{n_1}(c-z)]dc,\ c>\max(0,z).$$

We now make the following assumption.

Assumption 4.1. $q_1, q_2 \in \mathcal{Q} = \{q\colon q(t+c-a)/q(t+c) \uparrow c \text{ for } c > a,\ \forall t \geq 0\}$.

It then follows that the family of distributions of c in (4.4), generated by z for fixed t_1, t_2, has MLR property in c by the assumption $q_2 \in \mathcal{Q}$ and hence R is nondecreasing in z by the assumption $q_1 \in \mathcal{Q}$ [vide Lehmann (1959), p. 74]. This proves the following result.

Theorem 4.1. For testing $H_0\colon \theta_1 = \theta_2$ vs $H_1\colon \theta_1 > \theta_2$ in the model (4.1), the test with the critical region $z > k$ is UMPI in the class of conditional level-α tests, conditionally given the two ancillary statistics t_1 and t_2, whenever $q_1, q_2 \in \mathcal{Q}$.

Remark 4.1. It is clear from (4.2) that when both q_1 and q_2 are exponential, t_1 and t_2 can be ignored and the above test is UMPI unconditionally. For non-exponential q's, sufficiency-invariance reduces $(x_{(1)}, y_{(1)}, t_1, t_2)$ to $(x_{(1)} - y_{(1)}, t_1, t_2)$ and the above conditional argument is necessary unless $n_1 = n_2 = 1$ in which case t_1 and t_2 become vacuous.

Remark 4.2. If $q \in \mathcal{Q}$ is differentiable, $\mathcal{Q}$ is equivalent to

$$\tilde{\mathcal{Q}} = \{q\colon cq'(t+c)/q(t+c) \downarrow c,\ \forall t \geq 0\}. \tag{4.5}$$

An example of a $q \in \tilde{\mathcal{Q}}$ is provided by $q(u) = e^{-u^r}$, $r > 0$.

Remark 4.3. The null pdf of z is given by

$$dP^z_{H_0}/dz = \int_{c>\max(0,z)} \tilde{q}_1[c]\tilde{q}_2[c-z]dc \tag{4.6}$$

where $\tilde{q}_1(c) = \int_0^\infty q_1[c+t_1]t_1^{\frac{n_1-1}{2}} dt_1$, $\tilde{q}_2(c-z) = \int_0^\infty q_2[c-z+t_2]t_2^{\frac{n_2-1}{2}} dt_2$. Thus the null robustness of the test does not hold.

5. <u>Test of Equality of Two Scale Parameters (Non-Exponential Case)</u>.

Let $\underset{\sim}{X}$: $n_1 \times 1$ and $\underset{\sim}{Y}$: $n_2 \times 1$ be independent random vectors with pdfs (with respect to Lebesgue measure)

$$(5.1)\quad f_{\mu_1,\sigma_1}(\underset{\sim}{x}) = \sigma_1^{-n_1} q_1[\sum_1^{n_1}(x_i-\mu_1)/\sigma_1],\ f_{\mu_2,\sigma_2}(\underset{\sim}{y}) = \sigma_2^{-n_2} q_2[\sum_1^{n_2}(y_i-\mu_2)/\sigma_2]$$

$$x_i \geq \mu_1,\ y_i \geq \mu_2,\ \forall i,\ -\infty < \mu_1,\mu_2 < \infty,\ \sigma_1,\sigma_2 > 0.$$

We consider the problem of testing H_0: $\sigma_1 = \sigma_2$ vs H_1: $\sigma_1 > \sigma_2$ when μ_1 and μ_2 are unknown. The problem remains invariant under the group G of transformations: $x_i \to ax_i+c_1$, $y_i \to ay_i+c_2$, $\mu_1 \to a\mu_1+c_1$, $\mu_2 \to a\mu_2+c_2$, $\sigma_1 \to a\sigma_1$, $\sigma_2 \to a\sigma_2$, $-\infty < c_1,c_2<\infty$, $a>0$. Let $x_{(1)} = \min x_i$, $y_{(1)} = \min y_i$, $t_1 = \sum_1^{n_1}(x_i-x_{(1)})$, $t_2 = \sum_1^{n_2}(y_i-y_{(1)})$. Applying Wijsman's (1967) theorem, the ratio of the pdfs of a maximal invariant $Z = t_1/t_2$ under nonnull to null is obtained as

$$(5.2)\quad R = \frac{dP^Z_{H_1}}{dP^Z_{H_0}} = \frac{\sigma_2^{-n_2}\displaystyle\int_{c_1>\mu_1-ax_{(1)},\,c_2>\mu_2-ay_{(1)},\,a>0} q_1[\frac{at_1+n_1(ax_{(1)}+c_1-\mu_1)}{\sigma_1}]q_2[\frac{at_2+n_2(ay_{(1)}+c_2-\mu_2)}{\sigma_2}]a^{n_1+n_2-1}\,da\,dc_1\,dc_2}{\sigma_1^{-n_2}\displaystyle\int_{c_1>\mu_1-ax_{(1)},\,c_2>\mu_2-ay_{(1)},\,a>0} q_1[\frac{at_1+n_1(ax_{(1)}+c_1-\mu_1)}{\sigma_1}]q_2[\frac{at_2+n_2(ay_{(1)}+c_2-\mu_2)}{\sigma_1}]a^{n_1+n_2-1}\,da\,dc_1\,dc_2}$$

$$= \frac{\int_0^\infty \tilde{q}_1[\frac{at_1}{\sigma_1}]\tilde{q}_2[\frac{at_2}{\sigma_2}]\sigma_2^{-(n_2-1)}\,da}{\int_0^\infty \tilde{q}_1[\frac{at_1}{\sigma_1}]\tilde{q}_2[\frac{at_2}{\sigma_2}]\sigma_1^{-(n_2-1)}\,da},\quad \tilde{q}(x) = \int_0^\infty q[x+u]\,du$$

$$= \frac{\int_0^\infty \tilde{q}_1[az\delta]\tilde{q}_2[a]\,da}{\int_0^\infty \tilde{q}_1[az]\tilde{q}_2[a]\,da}\cdot \delta^{-(n_2-1)},\quad \delta = \frac{\sigma_2}{\sigma_1},\quad z = \frac{t_1}{t_2}$$

$$= \frac{\int_0^\infty \tilde{q}_1[a\delta]\tilde{q}_2[az^{-1}]da}{\int_0^\infty \tilde{q}_1[a]\tilde{q}_2[az^{-1}]da}\,\delta^{-(n_2-1)}$$

$$= \delta^{-(n_2-1)} E[\tilde{q}_1[a\delta]/\tilde{q}_1[a]]$$

where E above stands for expectation with respect to a having the pdf

(5.3) $$a \sim \tilde{q}_1[a]\tilde{q}_2[az^{-1}]/\int_0^\infty \tilde{q}_1[a]\tilde{q}_2[az^{-1}]da, \quad a > 0.$$

We now make the following assumption.

Assumption 5.1. $\tilde{q}_1, \tilde{q}_2 \in \tilde{\mathcal{Q}} = \{\tilde{q}: \tilde{q}[ad]/\tilde{q}[a]\uparrow a \text{ for } 0 < d < 1\}$.

It then follows (vide Lehmann (1959), p. 74) that the family (5.3) has MLR property in a and hence R is nondecreasing in z. Thus we have proved the following.

Theorem 5.1. For testing $H_0: \sigma_1 = \sigma_2$ vs. $\sigma_1 > \sigma_2$ in the model (5.1), the test with the critical region $z > k$ is UMPI whenever $\tilde{q}_1, \tilde{q}_2 \in \tilde{\mathcal{Q}}$.

Remark 5.1. If either μ_1 or μ_2 or both are known, t_1 and t_2 are suitably redefined and the assumption 5.1 is modified accordingly.

Remark 5.2. The null pdf of z is given by $z^{-(n_2+3)/2}\int_0^\infty \tilde{q}_1[a]\tilde{q}_2[az^{-1}]\, a^{\frac{n_1+n_2}{2}} da$.

Hence for specific q_1 and q_2 signigicance points of z are obtained from it.

Remark 5.3. If $\tilde{q}$ admits a derivative, $\tilde{\mathcal{Q}}$ is equivalent to

$$\tilde{\mathcal{Q}} = \{q: a\tilde{q}'(a)/\tilde{q}(a) \downarrow a > 0\}.$$

An example of a $\tilde{q} \in \tilde{\mathcal{Q}}$ is provided by $\tilde{q}(u) = e^{-u^r}$, $r > 0$.

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The problems of testing the equality of two location parameters (without the presence of scale parameters) and the equality of two scale parameters (with or without the presence of location parameters) based on two independent samples are considered. It is shown that, under some mild conditions on the underlying distributions, the standard optimum invariant tests derived under normality or exponentiality continue to be optimum for a wide class of distributions. However, the tests are not null robust. As a technical tool,

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(Abstract) continued...

Wijsman's (Fifth Berkeley Symposium, Vol. 1, 1967) representation theorem is used.

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